【Theorem 1】 The simple polygon with *n* sides, where *n* is an integer with $n \geq 3$, can be triangulated into *n*-2 triangles.

Proof:

Let *T(n)* be the statement that simple polygon with *n* sides can be triangulated into *n*-2 triangles

(1) Basis step: *T*(3) is true.

(2) Inductive step:

Assume that *T*(*j*) is true for all integers *j* with $3 \leq j \leq k$. We must show *T*(*k*+1) is true, that is that every simple polygon with *k*+1 sides can be triangulated into *k*-1 triangles.

Suppose that we have a simple polygon P with *k*+1 sides. By Lemma 1, P has an interior diagonal *ab*. *ab* splits P into two simple polygon Q, with *s* $(3 \leq s \leq k)$ sides, and R, with *t* $(3 \leq t \leq k)$ sides. (detail omitted.)

By strong induction, every simple polygon with n sides, where $n \geq 3$, can be Triangulated into n-2 triangles.

LAME'S Theorem **Let**
Then the number of divi
to find gcd (a, b) is less t
of decimal digits in b.

$r_0 = r_1 q_1 + r_2 \quad 0 \le r_2 < r_1$

$r_1 = r_2 q_2 + r_3 \quad 0 \le r_3 < r_2$

…

$r_{n-2} = r_{n-1} q_{n-1} + r_n \quad 0 \le r_n < r_{n-1}$

$r_{n-1} = r_n q_n$

Proof:

The number of divisions $n \le 5k$

From the above equations,

$q_1, q_2, \cdots, q_{n-1} \ge 1 \qquad q_n \ge 2 \quad \because r_n < r_{n-1}$

This implies

$$r_n \ge 1 = f_2$$

$$r_{n-1} \ge 2r_n \ge 2f_2 = f_3$$

$$r_{n-2} \ge r_{n-1} + r_n \ge f_3 + f_2 = f_4$$

$$r_{n-3} \ge r_{n-2} + r_{n-1} \ge f_3 + f_4 = f_5$$

$\dots$

$$r_2 \geq r_3 + r_4 \geq f_{n-1} + f_{n-2} = f_n$$

$$b = r_1 \geq r_2 + r_3 \geq f_n + f_{n-1} = f_{n+1}$$

$$\because f_{n+1} > \alpha^{n-1} \text{ for } n > 2$$

$$\therefore b > \alpha^{n-1}$$

$$\log_{10} b > (n-1)\log_{10} \alpha > \frac{n-1}{5}$$

$$n - 1 < 5 \cdot \log_{10} b < 5k$$

$$\therefore n \leq 5k$$

$r_0 = r_1 q_1 + r_2 \qquad 0 \leq r_2 < r_1$

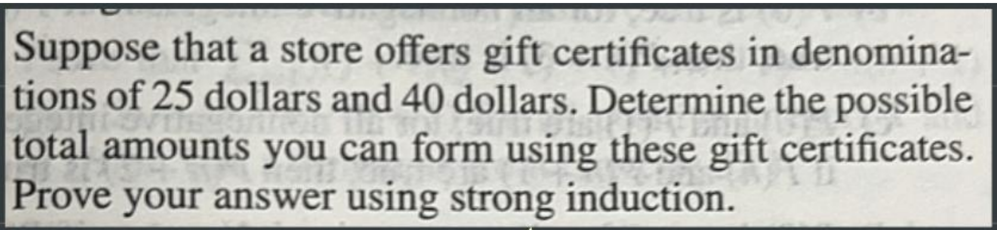
$r_1 = r_2 q_2 + r_3 \qquad 0 \leq r_3 < r_2$

$\dots$

$r_{n-2} = r_{n-1} q_{n-1} + r_n \qquad 0 \leq r_n < r_{n-1}$

$r_{n-1} = r_n q_n$

e.g (HW 5.2T8)



Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

① 寻找最大公因数，仅最大公因数$^{\nearrow k}$倍数可被表示，并将所有数除以最大公因数从小到大得 $x_1, x_2, \dots$

② 找 $y = k_1x_1 + k_2x_2 + \dots$ 的连续 x_1 个数，则 $kn\ (n \geq y_{x_1})$ 均可被表示. $\left(P(y) = P(y - x_1)\right)$

↑ Basis ↑ Inducive

prove by well-ordered property
< prove by contradiction

结合 ⇒ s1. 构造一个 well-ordered 的集合，根据 property 取出 least number k.

s2. 假定这个集合中没有满足条件的 k_0（即 k 不满足题设条件），推出 k 不是这个集合的 least number，故矛盾，所以假设不成立

⇒ 有 k_0 满足题设条件

〖Example 11〗 Use generating functions to solve the recurrence relation$a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6$ **with initial conditions**$a_0 = 20, a_1 = 60$**.**

Solution:

$$a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6 \qquad \times x^n$$

$$a_n x^n = 2a_{n-1}x^n + 3a_{n-2}x^n + 4^n x^n + 6x^n$$

$$\begin{tikzcd}
\sum_{n=2}^{\infty} a_n x^n \arrow[d] & = 2\sum_{n=2}^{\infty} a_{n-1}x^n \arrow[d] & + 3\sum_{n=2}^{\infty} a_{n-2}x^n \arrow[d] & + \sum_{n=2}^{\infty} 4^n x^n \arrow[d] & + 6\sum_{n=2}^{\infty} x^n \arrow[d] \\
G(x) - a_0 - a_1 x & 2x\sum_{n=1}^{\infty} a_n x^n \arrow[d] & 3x^2\sum_{n=0}^{\infty} a_n x^n \arrow[d] & \frac{1}{1-4x} - 1 - 4x & 6(\frac{1}{1-x} - 1 - x) \\
 & 2x(G(x) - a_0) & 3x^2 G(x) & &
\end{tikzcd}$$

$$(1-2x-3x^2)G(x)=\frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)}$$

$$G(x)=\frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)(1+x)(1-3x)}$$

$$\begin{array}{cccccccc}
= & \dfrac{16/5}{1-4x} & + & \dfrac{-3/2}{1-x} & + & \dfrac{31/20}{1+x} & + & \dfrac{67/4}{1-3x} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \dfrac{16}{5}\times 4^n & & -\dfrac{3}{2}\times 1^n & & \dfrac{31}{20}\times(-1)^n & & \dfrac{67}{4}\times 3^n
\end{array}$$

$$a_n=\frac{16}{5}\times 4^n-\frac{3}{2}+\frac{31}{20}\times(-1)^n+\frac{67}{4}\times 3^n$$

〖Example 12〗 Use generating functions to prove Pascal's identity $C(n,r) = C(n-1,r) + C(n-1,r-1)$ when *n* and *r* are positive integers with $r < n$.

Proof:

$$G(x) = (1+x)^n = \sum_{r=0}^{n} C(n,r)x^r$$

$$(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$$

$$\sum_{r=0}^{n} C(n,r)x^r = \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1}$$

$$= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r$$

$$1 + \sum_{r=1}^{n-1} C(n,r)x^r + x^n$$

$$= 1 + \sum_{r=1}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n-1} C(n-1,r-1)x^r + x^n$$

$$\sum_{r=1}^{n-1} \underline{C(n,r)}x^r = \sum_{r=1}^{n-1} [\underline{C(n-1,r) + C(n-1,r-1)}]x^r$$