COMP481 Review Problems
Turing Machines and (Un)Decidability
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Problems:

1. For each of the following languages, state whether each language is (I) recursive, (II) recursively enumerable but not recursive, or (III) not recursively enumerable. Prove your answer.

   - $L_1 = \{\langle M\rangle | M$ is a TM and there exists an input on which M halts in less than $|\langle M\rangle|$ steps$\}$.
   - $L_2 = \{\langle M\rangle | M$ is a TM and $|L(M)| \leq 3\}$.
   - $L_3 = \{\langle M\rangle | M$ is a TM and $|L(M)| \geq 3\}$.
   - $L_4 = \{\langle M\rangle | M$ is a TM that accepts all even numbers$\}$.
   - $L_5 = \{\langle M\rangle | M$ is a TM and $L(M)$ is finite$\}$.
   - $L_6 = \{\langle M\rangle | M$ is a TM and $L(M)$ is infinite$\}$.
   - $L_7 = \{\langle M\rangle | M$ is a TM and $L(M)$ is countable$\}$.
   - $L_8 = \{\langle M\rangle | M$ is a TM and $L(M)$ is uncountable$\}$.
   - $L_9 = \{\langle M_1, M_2\rangle | M_1$ and M_2 are two TMs, and $\varepsilon \in L(M_1) \cup L(M_2)\}$.
   - $L_{10} = \{\langle M_1, M_2\rangle | M_1$ and M_2 are two TMs, and $\varepsilon \in L(M_1) \cap L(M_2)\}$.
   - $L_{11} = \{\langle M_1, M_2\rangle | M_1$ and M_2 are two TMs, and $\varepsilon \in L(M_1) \setminus L(M_2)\}$.
   - $L_{12} = \{\langle M\rangle | M$ is a TM, M_0 is a TM that halts on all inputs, and $M_0 \in L(M)\}$.
   - $L_{13} = \{\langle M\rangle | M$ is a TM, M_0 is a TM that halts on all inputs, and $M \in L(M_0)\}$.
   - $L_{14} = \{\langle M, x\rangle | M$ is a TM, x is a string, and there exists a TM, M', such that $x \notin L(M) \cap L(M')\}$.
   - $L_{15} = \{\langle M\rangle | M$ is a TM, and there exists an input on which M halts within 1000 steps$\}$.
   - $L_{16} = \{\langle M\rangle | M$ is a TM, and there exists an input whose length is less than 100, on which M halts$\}$.
   - $L_{17} = \{\langle M\rangle | M$ is a TM, and M is the only TM that accepts $L(M)\}$.
   - $L_{18} = \{\langle k, x, M_1, M_2, \ldots, M_k\rangle | k$ is a natural number, x is a string, M_i is a TM for all $1 \leq i \leq k$, and at least $k/2$ TMs of $M_1, \ldots, M_k$ halt on $x\}$.
   - $L_{19} = \{\langle M\rangle | M$ is a TM, and $|M| < 1000\}$.
   - $L_{20} = \{\langle M\rangle | \exists x, |x| \equiv_5 1$, and $x \in L(M)\}$.
   - $L_{21} = \{\langle M\rangle | M$ is a TM, and M halts on all palindromes$\}$.
   - $L_{22} = \{\langle M\rangle | M$ is a TM, and $L(M) \cap \{a^{2^n} | n \geq 0\}$ is empty$\}$.
   - $L_{23} = \{\langle M, k\rangle | M$ is a TM, and $|\{w \in L(M) : w \in a^*b^*\}| \geq k\}$.
   - $L_{24} = \{\langle M\rangle | M$ is a TM that halts on all inputs and $L(M) = L'$ for some undecidable language $L'\}$.
   - $L_{25} = \{\langle M\rangle | M$ is a TM, and M accepts (at least) two strings of different lengths$\}$.
   - $L_{26} = \{\langle M\rangle | M$ is a TM such that both $L(M)$ and $\overline{L(M)}$ are infinite$\}$.
   - $L_{27} = \{\langle M, x, k\rangle | M$ is a TM, and M does not halt on x within k steps$\}$.
   - $L_{28} = \{\langle M\rangle | M$ is a TM, and $|L(M)|$ is prime$\}$.
   - $L_{29} = \{\langle M\rangle |$ there exists $x \in \Sigma^*$ such that for every $y \in L(M)$, $xy \notin L(M)\}$.
   - $L_{30} = \{\langle M\rangle |$ there exist $x, y \in \Sigma^*$ such that either $x \in L(M)$ or $y \notin L(M)\}$.
   - $L_{31} = \{\langle M\rangle |$ there exists a TM M' such that $\langle M\rangle \neq \langle M'\rangle$ and $L(M) = L(M')\}$.
   - $L_{32} = \{\langle M_1, M_2\rangle | L(M_1) \leq_m L(M_2)\}$.

- $L_{33} = \{\langle M \rangle | M$ does not accept any string w such that 001 is a prefix of $w\}$.
- $L_{34} = \{\langle M, x \rangle | M$ does not accept any string w such that x is a prefix of $w\}$.
- $L_{35} = \{\langle M, x \rangle | x$ is prefix of $\langle M \rangle\}$.
- $L_{36} = \{\langle M_1, M_2, M_3 \rangle | L(M_1) = L(M_2) \cup L(M_3)\}$.
- $L_{37} = \{\langle M_1, M_2, M_3 \rangle | L(M_1) \subseteq L(M_2) \cup L(M_3)\}$.
- $L_{38} = \{\langle M_1 \rangle |$ there exist two TMs M_2 and M_3 such that $L(M_1) \subseteq L(M_2) \cup L(M_3)\}$.
- $L_{39} = \{\langle M, w \rangle | M$ is a TM that accepts w using at most $2^{|w|}$ squares of its tape$\}$.

2. If $A \leq_m B$ and B is a regular language, does that imply that A is a regular language?

3. Recall the language $A_{TM} = \{\langle M, w \rangle | M$ is a TM, and M accepts $w\}$. Consider the language

$$J = \{w | w = 0x \text{ for some } x \in A_{TM} \text{ or } w = 1y \text{ for some } y \in \overline{A_{TM}}\}.$$

   (a) Show that J is not in RE.
   (b) Show that $\overline{J}$ is not in RE.
   (c) Show that $J \leq_m \overline{J}$.

4. Show that if a language A is in RE and $A \leq_m \overline{A}$, then A is recursive.

5. A language L is **RE-Complete** if:

   - $L \in RE$, and
   - $L' \leq_m L$ for all $L' \in RE$.

   Recall the following languages:

$$L_{\Sigma^*} = \{\langle M \rangle |\ L(M) = \Sigma^*\}$$
$$HP = \{\langle M, w \rangle |\ M \text{ halts on } w\}$$

   (a) Is L_{Σ^*} RE-Complete or not? Prove your answer.
   (b) Is HP RE-Complete or not? Prove your answer.

6. Let L_1, L_2 be two decidable languages, and let L be a language such that $L_1 \subseteq L \subseteq L_2$. Is L decidable or not? Prove your answer.

7. Let L be a language RE. Show that $L' = \{x | \exists y : (x, y) \in L\}$ is also RE.

8. Prove or disprove: there exists an undecidable unary language (a unary language is a subset of 1^*).

9. PROBLEM FORMULATION.

   (a) Consider the problem of testing whether a TM M on an input w ever attempts to move its head left when its head is on the leftmost tape cell. Formulate this problem as a language and show that it is undecidable.
   (b) Consider the problem of testing whether a TM M on an input w ever attempts to move its head left at any point during its computation on w. Formulate this problem as a language and show that it is decidable.

10. Let A and B be two disjoint languages. We say that language C **separates** A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two disjoint co-RE languages are separable by some decidable language.

11. Suppose there are four languages A, B, C, and D. Each of the languages may or may not be recursively enumerable. However, we know the following about them:

- There is a reduction from A to B.
- There is a reduction from B to C.
- There is a reduction from D to C.

Below are four statements. Indicate whether each one is

(a) CERTAIN to be true, regardless of what problems A through D are.
(b) MAYBE true, depending on what A through D are.
(c) NEVER true, regardless of what A through D are.

Please, justify your answer!

(a) A is recursively enumerable but not recursive, and C is recursive.
(b) A is not recursive, and D is not recursively enumerable.
(c) If C is recursive, then the complement of D is recursive.
(d) If C is recursively enumerable, then $B \cap D$ is recursively enumerable.

12. Recall the following definition: A grammar G computes a function f iff for all $u, v \in \Sigma^*$,

$$SuS \Rightarrow_G{}^* v \text{ iff } f(u) = v.$$

For each of the following functions, show a grammar that computes it. In the functions $f_1, \ldots, f_4$, both n and $f(n)$ are unary representations of natural numbers. For functions $f_5, \ldots, f_8$, the input/output alphabet is specified.

- $f_1(n) = 3n + 5$.
- $f_2(n) = \begin{cases} 1 & if \quad n \equiv 0 \ (mod\ 3) \\ 11 & if \quad n \equiv 1 \ (mod\ 3) \\ 111 & if \quad n \equiv 2 \ (mod\ 3) \end{cases}$
- $f_3(n) = n - 1$.
- $f_4(n) = n/2$. Assume n is even.
- $f_5(w) = ww$, where $w \in \{a, b\}^*$.
- $f_6 = w'$, where $w \in \{a, b\}^*$, and w' is obtained from w by replacing the a's by b's and b's by a's. For example, $f_6(aaba) = bbab$.
- $f_7(a_1 a_2 \ldots a_k) = a_1 a_1 a_2 a_2 \ldots a_k a_k$, where each a_i is in the alphabet $\{a, b\}$. For example, $f_7(aaba) = aaaabbaa$.
- $f_8(w) = \begin{cases} f_6(w) & if \quad the\ rightmost\ symbol\ of\ w\ is\ a \\ f_7(w) & if \quad the\ rightmost\ symbol\ of\ w\ is\ b \end{cases}$ $(\Sigma = \{a, b\})$.

13. Show that the following languages are recursive.

- $L_{40} = \{\langle M \rangle | M$ is a DFA and $L(M)$ is finite$\}$.
- $L_{41} = \{\langle M \rangle | M$ is a DFA and $L(M) = \Sigma^*\}$.
- $L_{42} = \{\langle M, x \rangle | M$ is a DFA and M accepts $x\}$.
- $L_{43} = \{\langle M, x \rangle | M$ is a DFA and M halts on $x\}$.
- $L_{44} = \{\langle G \rangle | G$ is a CFG and $L(G) = \emptyset\}$.
- $L_{45} = \{\langle M \rangle | M$ is a DFA and M accepts some string of the form ww^R for some $w \in \{a, b\}^*\}$.

14. Prove that each of the following languages are not context-free, and write unrestricted grammars that generate them.

- $L_{46} = \{x\sharp w|\ x, w \in \{a,b\}^*$ and x is a substring of $w\}$.
- $L_{47} = \{w \in \{a,b,c\}^*|\ \sharp_a(w) \geq \sharp_b(w) \geq \sharp_c(w)\}$.
- $L_{48} = \{a^n b^n c a^n b^n | n > 0\}$.
- $L_{49} = \{a^n b^{2n} c^{3n} | n \geq 0\}$.
- $L_{50} = \{a^n b^{n+m} c^m d^n | m, n \geq 0\}$.
- $L_{51} = \{w \in \{1\}^* | w$ is the unary encoding of 2^k for some $k > 0\}$.

15. Let L_{52} be the language containing only the single string s, where

$$s = \begin{cases} 0 & if \quad God\ does\ not\ exist \\ 1 & if \quad God\ exists \end{cases}$$

Is L_{52} decidable? Why or why not? (Note that the answer does not depend on your religious convictions.)